Let $p=2rs$; $q=r^2+s^2$. Then $x=\dfrac{4r^2s^2}{r+s}$; $y=\dfrac{(r^2+s^2)^2}{r+s}$.

Let $r=k+l$; $s=k-l$. Then $x=\dfrac{2(k^2-l^2)^2}{k}$; $y=\dfrac{2(k^2+l^2)^2}{k}$.

Let $l=\alpha k$. Then $x=2k^3(1-\alpha^2)^2$; $y=2k^3(1+\alpha^2)^2$.

Now $a=p^3=8r^3s^3=8(k^2-l^2)^3=8k^6(1-\alpha^2)^3$, and $b=q^3=(r^2+s^2)^3=8(k^2+l^2)^3=8k^6(1+\alpha^2)^3$, where α and k are integers.

PROBLEMS.

53. Proposed by A. H. BELL, Box 184, Hillsboro, Illinois.

Given $x^2-114\frac{1}{4}y^2=\mp3$ to find the least values of x and y in integers.

54. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

In the expression $2x^2-2ax+b^2$, find two series of values for x in integral terms of a and b.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

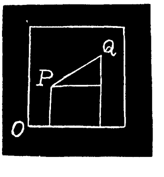
SOLUTIONS OF PROBLEMS.

35. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the chance that the distance of two points within a square shall not exceed a side of the square. [From *Byerly's Integral Calculus.*]

I. Solution by ALWYN C. SMITH, The University of Colorado, Boulder, Colorado.

a is one side of the square; P and Q the two points; (x, y) the point P with O for origin; and r and ϕ the polar coordinates of Q, with P as origin. Then the favorable cases are

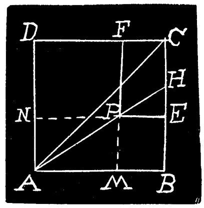


$$4\int_0^{\frac{1}{2}\pi}\int_0^a\int_0^{a-r\sin\phi}\int_0^{a-r\cos\phi}dxdyrdrd\phi=a^4(\pi-\tfrac{13}{6}).$$

All the cases $=a^2.a^2=a^4$. Therefore, $p=\pi-\frac{13}{6}$.

II. Solution by J. M. COLAW, A. M., Principal of High School, Monterey, Virginia.

Let a=a side of the square $ABCD$, and join A with any point P within the given square. Then as AP represents the distance and direction of the second point from the first, the area of the rectangle $PECF$ represents the number of ways the two points can be taken.



Let $AP=x$, $AH=x'$, and $\angle PAB=\theta$.

When $x'=a\sec\theta$, $PF=a-x\sin\theta$, $PE=a-x\cos\theta$.

$\therefore$ Area $PECF=(a-x\sin\theta)(a-x\cos\theta)$.

Hence the required chance is

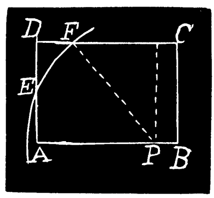
$$p_1=\frac{\int_0^{\frac{1}{4}\pi}\int_0^a (a-x\sin\theta)(a-x\cos\theta)xd\theta dx}{\int_0^{\frac{1}{4}\pi}\int_0^{x'} (a-x\sin\theta)(a-x\cos\theta)xd\theta dx}$$

$$=\frac{8}{a^4}\int_0^{\frac{1}{4}\pi}\int_0^a (a-x\sin\theta)(a-x\cos\theta)xd\theta dx$$

$$=\tfrac{2}{3}\int_0^{\frac{1}{4}\pi}(6-4\sin\theta-4\cos\theta+3\sin\theta\cos\theta)d\theta=\pi-\tfrac{13}{6}.$$

III. Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.

Take a rectangle $ABCD$, with sides $AB=b$, and $BC=a$, such that a is not greater than b; and consider the chance that the proposed distance shall exceed b. Let N be the number of favorable cases; then if a be increased infinitesimally, dN will be the number of new cases introduced by placing each point in turn on the differential slice along b while the other one traverses the mixtilinear area DEF.



That is, taking AP equal to x,

$$dN=4\Big[\int_{\sqrt{b^2-a^2}}^{b}(ax-\frac{a}{2}\sqrt{b^2-a^2}-\frac{x}{2}\sqrt{b^2-x^2}$$

$$-\frac{b^2}{2}\sin^{-1}\frac{x}{b}+\frac{b^2}{2}\cos^{-1}\frac{a}{b})\,dx\Big]da$$

$$=2[2ab-ab\sqrt{b^2-a^2}-\frac{a^3}{3}-\frac{\pi b^3}{2}+b^3\cos^{-1}\frac{a}{b}]da\text{ ; and,}$$

$$N=2[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^4}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}\,]+C.$$

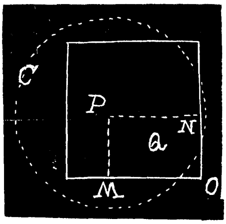
Since $N=0$ when $a=0$, $C=\frac{4b^4}{3}$; and,

$$N=2[a^2b^2+\frac{b}{3}\sqrt{(b^2-a^2)^3}-\frac{a^2}{12}-\frac{\pi ab^3}{2}+ab^3\cos^{-1}\frac{a}{b}-b^3\sqrt{b^2-a^2}+\frac{2b^4}{3}].$$

If now $a=b$, $N=b^4(\frac{19}{6}-\pi)$; and the whole number of cases$=b^4$. Hence the chance that the proposed distance shall exceed b is $\frac{19}{6}-\pi$; therefore, the chance that it will not exceed b is $\pi-\frac{13}{6}$.

IV. **Solution by LEWIS NEIKIRK, Senior in the University of Colorado, Boulder, Colorado.**

Let a be one side of the square, and O the origin. With center O and radius a describe a quadrant. Let P any point within the square (x, y) be one point, and Q be the other point. With center P and radius a describe the circle C. Now Q may be anywhere within the area common to this circle and the square. The favorable cases may then be found by confining Q within the rectangle xy while P traverses the entire square, and then taking four times the result. Hence,



$$p=\frac{4}{a^4}\left\{\int_0^a\int_0^{\sqrt{a^2-y^2}}xydxdy+\tfrac{1}{2}\int_0^a\int_{\sqrt{a^2-y^2}}^a[x\sqrt{a^2-x^2}\right.$$

$$\left.+y\sqrt{a^2-y^2}+a^2\sin^{-1}\frac{x}{a}-a^2\cos^{-1}\frac{y}{a}]dxdy\right\}=\pi-\frac{13}{6}.$$

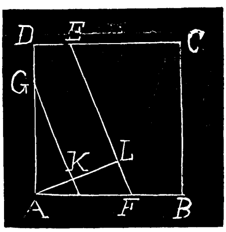
V. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

This problem affords a splendid test of the correctness of the general value for any convex area as demonstrated in problem 25, page 281, September-October MONTHLY.

Let AK, $AL=p$, $\angle LAB=\theta$, EF, $GH=C$.

For EF, $C=a\sec\theta$; the limits of p are $a\sin\theta$ to $a\cos\theta$.

For GH, $C=p\sec\theta\operatorname{cosec}\theta$; the limits of p are $a\sin\theta\cos\theta$ to $a\sin\theta$. The limits of θ are 0 to $\frac{1}{4}\pi$.



From problem 25,

$$\Delta=\frac{1}{3A^2}\int\int(C^3-3a^2C+2a^3)d\theta dp.$$

$$\therefore\ \Delta=\frac{8}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta\cos\theta}^{a\sin\theta}(p^3\sec^3\operatorname{cosec}^3\theta-3a^2p\sec\theta\operatorname{cosec}\theta+2a^3)d\theta dp$$

$$+\frac{4}{3a^4}\int_0^{\frac{1}{4}\pi}\int_{a\sin\theta}^{a\cos\theta}(a^3\sec^3\theta-3a^3\sec\theta+2a^3)d\theta dp.$$

$$\triangle=\tfrac{2}{3}\int_0^{\frac{1}{4}\pi}(\tan\theta\sec^2\theta-3\sin\theta\cos\theta-6\tan\theta+8\sin\theta)d\theta$$

$$+\tfrac{4}{3}\int_0^{\frac{1}{4}\pi}(\sec^2\theta-\tan\theta\sec^2\theta-3+3\tan\theta+2\cos\theta-2\sin\theta)d\theta.$$

$\therefore\ \triangle=\frac{19}{6}-\pi.\quad p=1-\triangle=\pi-\frac{13}{6}=$required chance.

PROBLEMS.

44. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

What is the average length of all the chords that may be drawn from one extremity of the major axis of an ellipse if they are drawn at equal angular intervals?

45. Proposed by J. C. WILLIAMS, Boston, Massachusetts.

At the end of the fifth inning the base ball score stands 7 to 9. What is the probability of winning for either team?

46. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

Four men starting from random points on the circumference of a circular field and traveling at different rates, take random straight courses across it; find the chance that at least two of them will meet.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

38. Proposed by S. H. WRIGHT, M. D., A. M., Ph. D., Penn Yan, New York.

In latitude $42°30'$ north$=\lambda$, at what angle with the horizon will the sun rise, its declination$=22°$ north$=\delta$?

I. Solution by the PROPOSER.

Let BA be a portion of the equator, $CA=\delta$, a portion of a meridian passing through the sun at C when rising, and describing a small-circle arc CE, parallel with BA, and let BC be a portion of the horizon. Then the angles ECA, and BAC, each$=90°$, because meridians cut the equator and circles of declination at right angles. Now $CBA=90°-\lambda$, then $\sin BCA=\sin\lambda\sec\delta=\cos BCE$. $\therefore\ BCE=43°13'37''=$required angle.